SOLUTION OF THE BASIC PROBLEM OF EXTERIOR BALLISTICS

(RESHENIE OSNOVNOI ZADACHI VNESHNEI BALLISTIKI)

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An approximate method of analytic solution of certain nonlinear problems is presented. The applicability of the method is demonstrated with the example of the solution of the problem concerning the motion of a heavy particle of variable mass, projected at an angle to the horizon. Another example contains the analytic solution of the basic problem of exterior ballistics of a projectile in the most general formulation.

1. Approximate method of integration of certain nonlinear differential equations. Let a system of ordinary nonlinear differential equations be given in the form

$$\begin{aligned}
\frac{dy_1}{dx} &= f_1(x, y_1, y_2, y_3, \ldots, y_n)\\
\frac{dy_2}{dx} &= f_2(x, y_1, y_3, \ldots, y_n)\\
\frac{dy_3}{dx} &= f_3(x, y_1, y_4, \ldots, y_n)\\
&\ldots\ldots\ldots\ldots\ldots\\
\frac{dy_{n-1}}{dx} &= f_{n-1}(x, y_1, y_n)\\
\frac{dy_n}{dx} &= f_n(x, y_1)
\end{aligned} \qquad (1.1)$$

The solution of the system has to satisfy the initial conditions

$$x = x_0, \qquad y_i = y_{i0}, \qquad y_i' = y_{i0}' \qquad (i = 1, \ldots, n) \qquad (1.2)$$

in a certain closed region D of variables x, y_i, where f_i are continuous, together with their second derivatives.

If the quadratures of both sides of the last $n-1$ equations are taken, and the found values $y_n, y_{n-1}, y_{n-2}, \ldots, y_2$, which satisfy the conditions (1.2), are substituted into each preceding equation of the system (1.1), then we find (from the first equation of this system which we shall call the fundamental one) for y_1 a functional relationship of the form

$$\frac{dy_1}{dx} - F[y_1] = 0 \tag{1.3}$$

where $F[y_1]$ designates a known function of the integral operators of y_1. We shall show how the solution of equation (1.3) may be found by using a certain modification of Chaplygin algorithm for an ordinary differential equation [1,2].

Let $\phi[y_1]$ be the left-hand side of equation (1.3). Assume that the functions y_{0-} and y_{0+} are found in the region D, such that

$$y_{0^-}(x_0) = y_{0^+}(x_0) = y_{10}, \qquad \Phi[y_{0^-}] \leqslant 0 \leqslant \Phi[y_{0^+}] \qquad (x \geqslant x_0) \tag{1.4}$$

We introduce Δy_- and Δy_+ through the equations

$$y_1 = y_{0^-} - \Delta y_-, \qquad y_1 = y_{0^+} - \Delta y_+ \tag{1.5}$$

Then, from (1.3), we obtain the equations

$$\frac{d(\Delta y_-)}{dx} + F[y_1] - F[y_{0^-}] - \Phi[y_{0^-}] = 0 \tag{1.6}$$

$$\frac{d(\Delta y_+)}{dx} + F[y_1] - F[y_{0^+}] - \Phi[y_{0^+}] = 0$$

Let us consider the difference $F[y_1] - F[y_{0-}]$. A finite increment of the functional operator may be represented in the form of a series of its variation with suitable factors. For example, limiting ourselves to just two terms of the series, we may write

$$F[y_1] - F[y_{0\_}] = \delta F y_{0^-} y_{0^+} + \tfrac{1}{2}\delta^2 F y_0 y_{0^-} \tag{1.7}$$

Here

$$\delta F y_{0^-} y_{0^+} = -\frac{d}{d\varepsilon} F[y_{0^-} + \varepsilon(y_{0^+} - y_{0^-})]\Big|_{\varepsilon=0} \frac{\Delta y_-}{y_{0^+} - y_{0^-}}$$

$$\delta^2 F y_0 y_{0^-} = \frac{d^2}{d\varepsilon^2} F[y_{0^-} + \varepsilon(y_0 - y_{0^-})]\Big|_{\varepsilon=0} \left(\frac{\Delta y_-}{y_{0^+} - y_{0^-}}\right)^2$$

In addition to this, the differential $F[y_1] - F[y_{0+}]$ may be also represented in the form

$$F[y_1] - F[y_{0^+}] = -\frac{F[y_{0^+}] - F[y_{0^-}]}{y_{0^+} - y_{0^-}} \Delta y_+ - \beta \tag{1.8}$$

Let the functions y_{0+} and y_{0-}, which we shall call the supporting ones, be already so close to each other, that for any y_0 in the interval $[y_{0+}, y_{0-}]$ and arbitrary $x \geqslant x_0$, the inequalities are fulfilled

$$\frac{1}{2}\delta^2 F y_0 y_{0^-} = \alpha \geqslant 0, \qquad \beta \geqslant 0 \tag{1.9}$$

Substituting the increments (1.7) and (1.8) into (1.6), we rewrite them in the form

$$\frac{d(\Delta y_-)}{dx} - p_-\Delta y_- - \Phi[y_{0\_}] = -\alpha, \quad \frac{d(\Delta y_+)}{dx} - p_+\Delta y_+ - \Phi[y_{0^+}] = \beta \tag{1.10}$$

where

$$p_- = \frac{d}{d\varepsilon} F[y_{0\_} + \varepsilon(y_{0^+} - y_{0^-})]\Big|_{\varepsilon=0} \frac{1}{y_{0^+} - y_{0^-}}, \qquad p_+ = \frac{F[y_{0^+}] - F[y_{0^-}]}{y_{0^+} - y_{0^-}}$$

If in the expression (1.9) the sign of the inequalities is opposite, the p_+ and p_- in (1.10) have to be interchanged.

Let us find the integrals of the linear differential equations:

$$\frac{d(\Delta y_{0^-})}{dx} - p_-\Delta y_{0^-} - \Phi[y_{0^-}] = 0, \qquad \frac{d(\Delta y_{0^+})}{dx} - p_+\Delta y_{0^+} - \Phi[y_{0^+}] = 0 \tag{1.11}$$

which vanish for $x = x_0$.

They will be:

$$\Delta y_{0^-} = \exp\left(\int_{x_0}^{x} p_-\,dx\right)\int_{x_0}^{x} \Phi[y_{0\_}]\exp\left(-\int_{x_0}^{x} p_-\,dx\right)dx$$

$$\Delta y_{0^+} = \exp\left(\int_{x_0}^{x} p_+\,dx\right)\int_{x_0}^{x} \Phi[y_{0^+}]\exp\left(-\int_{x_0}^{x} p_+\,dx\right)dx \tag{1.12}$$

Substitution of zero in place of Δy_- and Δy_{0+} into the left-hand sides of equations (1.11) yields a result which corresponds to inequalities (1.4), wherefrom, according to Chaplygin's theorem [1], it follows

$$\Delta y_{0^-} \leqslant 0 \leqslant \Delta y_{0^+} \tag{1.13}$$

Replacement of Δy_{0-} and Δy_{0+} in the left-hand sides of equations (1.11) by Δy_- and Δy_+ using (1.10) gives the result $-\alpha < 0$ and $\beta > 0$, respectively, wherefrom, on the basis of this same theorem, follow the inequalities

$$\Delta y_- \leqslant \Delta y_{0\_}, \qquad \Delta y_+ \geqslant \Delta y_{0^+} \tag{1.14}$$

If the new approximations are designated by

$$y_{1^-} = y_{0^-} - \Delta y_{0^-}, \qquad y_{1^+} = y_{0^+} - \Delta y_{0^+} \tag{1.15}$$

then, from (1.5), (1.13) and (1.14), the inequalities follow

$$y_{0-} \leqslant y_{1-} \leqslant y_1 \leqslant y_{1+} \leqslant y_{0+}$$

The indicated method of iteration permits the finding of the solution of equation (1.3). The values of $y_2, \ldots, y_n$ are then determined by quadratures.

Fixing attention on the solution in first approximation, it is convenient, in practical application of the method, to choose one of the supporting functions in the form of the integral of the fundamental differential equation of the system (1.1), in which the right-hand side is altered only slightly:

$$\frac{dy_0^*}{dx} = f_1^*(x, y_0^*)$$

but such that its solution is determined analytically. In choosing this first supporting function, the main factors of the concrete problem should be taken into account.

As a second supporting function it is possible to take the y_{0*} - function, which is determined by the method of successive approximations from:

$$y_{0*} = y_0 + \int_{x_0}^{x} F[y_0^*]\, dx \qquad (1.16)$$

Then, instead of p_+, we find p:

$$p = \frac{F[y_0^*] - F[y_{0*}]}{y_0^* - y_{0*}} \qquad (1.17)$$

and the first approximation to the solution of equation (1.3) will be in the form:

$$y_{1*} = y_0^* - \exp\left(\int_{x_0}^{x} p\, dx\right) \int_{x_0}^{x} \Phi[y_0^*] \exp\left(-\int_{x_0}^{x} p\, dx\right) dx \qquad (1.18)$$

where

$$\Phi[y_0^*] = \frac{dy_0^*}{dx} - F[y_0^*] \qquad (1.19)$$

The examples given below (Sections 2 and 3) show that the first approximations yield already a high degree of accuracy.

2. Generalization of the problem of K.E. Tsiolkovskii for the case of curvilinear motion. Let a heavy material particle, whose mass changes in accordance,with some law $M = M_0 f(t)$, be projected at an angle θ_0 to the horizon with velocity v_0.

We shall assume that the gravity field is homogeneous and that the earth is plane and at rest, together with the atmosphere. During motion the particle is subjected to the force of gravity G, to the drag Q and to a force R, which is determined by the separation and association of mass particles. Let the direction of force R coincide with the direction of the tangent to the trajectory; its magnitude may depend on the elevation y, the speed of flight v, and the time of motion t of the particle. Thus

$$G = M_0 g f(t), \qquad Q = Q(y, v), \qquad R = R(t, y, v)$$

are known functions. For brevity we write

$$\frac{G}{M} = g, \qquad \frac{Q}{M} = q(t, y, v) \qquad \frac{R}{M} = r(t, y, v) \tag{2.1}$$

The equations of motion [3] of the particle are of the form

$$M\dot{v} = R - G\sin\theta - Q, \qquad Mv\dot{\theta} = -G\cos\theta \tag{2.2}$$

$$\dot{x} = v\cos\theta, \qquad \dot{y} = v\sin\theta \tag{2.3}$$

Substituting the variable θ by a new variable ϕ by means of the relation

$$\varphi = \ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \tag{2.4}$$

and using the notations (2.1), the system of equations (2.2) and (2.3) is reduced to the form

$$\dot{v} = r(t, y, v) - g \operatorname{th}\varphi - q(t, y, v)$$

$$\dot{\varphi} = -\frac{g}{v}, \qquad \dot{x} = \frac{v}{\operatorname{ch}\varphi}, \qquad \dot{y} = v \operatorname{th}\varphi \tag{2.5}$$

The integration of equations (2.5) will be carried out using the method described in the preceding section. From the last three equations we find

$$\varphi = \varphi_0 - g\int_{t_0}^{t} \frac{dt}{v}, \qquad x = x_0 + \int_{t_0}^{t} \frac{v}{\operatorname{ch}\varphi} dt, \qquad y = y_0 + \int_{t_0}^{t} v \operatorname{th}\varphi \, dt \tag{2.6}$$

If y from (2.6) is substituted into the first equation (2.5) and then the value of $\phi = \phi[v]$, also found from (2.6), is substituted as well, then we find for v a functional equation of the type (1.3):

$$\frac{dv}{dt} - F[v] = 0, \qquad F[v] = r[v] - g \operatorname{th}\varphi[v] - q[v] \tag{2.7}$$

Let us study the case when $Q = 0$ and $R = -vM_0 f$, that is the atmosphere is absent; V is the relative velocity of rejected particles [3]. Then the system of equations of motion (2.5) and expression (2.7) takes the form:

$$\dot{v} = -V\frac{\dot{f}}{f} - g \operatorname{th}\varphi, \qquad \dot{\varphi} = -\frac{g}{v}, \qquad \dot{x} = \frac{v}{\operatorname{ch}\varphi}, \qquad \dot{y} = v \operatorname{th}\varphi \tag{2.8}$$

$$F[v] = -V\frac{\dot{f}}{f} - g \operatorname{th}\varphi[v]$$

Let us choose as supporting functions the integrals of the equations

$$\dot{v}_- = -V\frac{\dot{f}}{f} - g \operatorname{th}\varphi_0, \qquad \dot{v}_+ = -V\frac{\dot{f}}{f} - g \operatorname{th}\varphi_{k-} \qquad \left(\varphi_{k-} = \varphi_0 - g\int_{t_0}^{t_k} \frac{dt}{v_-}\right)$$

where t_k is the time at the end of the active part.

Thus, the selected supporting functions v_- and v_+ correspond to the motion of the particle along straight lines, inclined with respect to the horizon at an angle θ_0 (initial angle of inclination of the velocity vect and angle θ_k (angle which is smaller than the angle of inclination of the velocity vector at the end of the active part), respectively. They are determined by the well-known formulas of Tsiolkovskii [3]. Then

$$\Phi[v_-] = -g\{\text{th}\,\varphi_0 - \text{th}\,\varphi[v_-]\} \leqslant 0,$$

$$\Phi[v_+] = g\{\text{th}\,[v_+] - \text{th}\,\varphi_{k-}\} \geqslant 0$$

that is, inequalities (1.4) are fulfilled.

Condition (1.9) is verified by the calculation of

$$\left.\frac{d^2F}{d\varepsilon^2}\right|_{\varepsilon=0} = \frac{2g^2}{\text{ch}^2\varphi}\left\{\int_{t_0}^{t}\frac{(v-v_-)^2}{v_-^3}\,dt + g\,\text{th}\,\varphi_-\left(\int_{t_0}^{t}\frac{v-v_-}{v_-^2}\,dt\right)^2\right\}$$

The first approximations are determined by formulas (1.15) and (1.12). Thus

$$v_{1-} = v_- - \exp\left(\int_{t_0}^{t}p_-dt\right)\int_{t_0}^{t}\Phi[v_-]\exp\left(-\int_{t_0}^{t}p_-\,dt\right)dt$$

$$v_{1+} = v_+ - \exp\left(\int_{t_0}^{t}p_+dt\right)\int_{t_0}^{t}\Phi[v_+]\exp\left(-\int_{t_0}^{t}p_+dt\right)dt$$

where

$$p_- = -\frac{g^2}{(v_+ - v_-)\,\text{ch}^2\varphi[v_-]}\int_{t_0}^{t}\frac{v_+ - v_-}{v_-^2}\,dt$$

$$p_+ = -\frac{\text{th}\,\varphi[v_+] - \text{th}\,\varphi[v_-]}{(\text{th}\,\varphi_0 - \text{th}\,\varphi_{k-})(t - t_0)}$$

The solution may be taken as one-half the sum of these approximations: $v_* = \frac{1}{2}(v_{1+} + v_{1-})$, whereby the error will not be larger than one-half their difference $\gamma = \frac{1}{2}(v_{1+} - v_{1-})$. Subsequently, ϕ, x and y are determined by formulas (2.6).

Let us consider a concrete example. Let us assume that the mass of the particle changes in accordance with the linear law $f = 1 - \beta t$, and the constant parameters and the initial conditions are as follows:

$$\beta = 0.01 \text{ sec}^{-1}, \qquad V = 2290 \text{ m sec}^{-1},$$

$$t_0 = x_0 = y_0 = 0, \quad v_0 = 100 \text{ m sec}^{-1}, \quad \theta_0 = 75^0$$

The results of the calculations are given in Table 1 and are illustrated in Fig. 1.

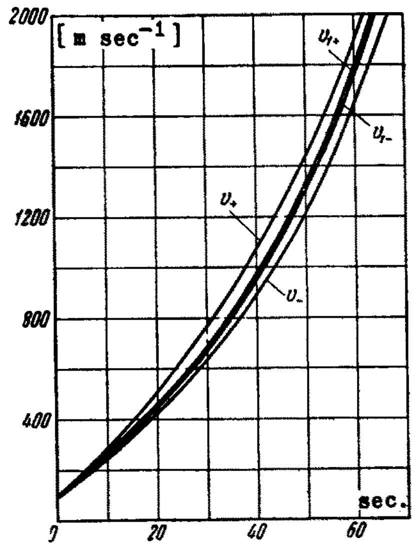

Fig. 1.

TABLE 1

t sec	v_- m sec^{-1}	v_+ m sec^{-1}	v_{1-} m sec^{-1}	v_{1+} m sec^{-1}	v_* m sec^{-1}	γ m sec^{-1}	γ %
0	100	100	100	100	100	0	0
5	170	192	171	171	171	0	0
10	246	290	250	250	250	0	0
15	332	398	341	341	341	0	0
20	423	511	438	438	438	0	0
30	632	764	666	666	666	0	0
40	891	1068	949	951	950	1	0.1
50	1214	1434	1301	1304	1303	2	0.2
60	1632	1896	1751	1759	1755	4	0.2
70	2207	2516	2361	2368	2365	4	0.2

As may be seen from Table 1, the one-half difference of first approximations, the upper and the lower, does not exceed 0.2 per cent.

Since the first approximations from above and below are very close, it is justified to limit one's attention in the solution of this problem to first approximation from one side only. Formulas (1.17) to (1.19) will be:

$$v_- = v_0 - V\ln f - g\,\mathrm{th}\,\varphi_0 (t - t_0), \qquad \varphi_- = \varphi_0 - g\int_{t_0}^{t} \frac{dt}{v_-}$$

$$v_+ = v_0 - V\ln f - g\,\mathrm{th}\,\varphi_{k-} (t - t_0), \qquad \varphi_+ = \varphi_0 - g\int_{t_0}^{t} \frac{dt}{v_+} \tag{2.9}$$

$$p = -\frac{\mathrm{th}\,\varphi_+ - \mathrm{th}\,\varphi_-}{(\mathrm{th}\,\varphi_0 - \mathrm{th}\,\varphi_{k-})(t - t_0)}, \quad \varphi_{k-} = \varphi_-(t_k)$$

$$v_* = v_- - g\exp\left(\int_{t_0}^{t} p\,dt\right)\int_{t_0}^{t} (\mathrm{th}\,\varphi_- - \mathrm{th}\,\varphi_0)\exp\left(-\int_{t_0}^{t} p\,dt\right)dt$$

Formulas (2.9) were also verified by means of comparison with the exact solution of equations (2.8), which can be obtained only for the case of the indicated law of loss of mass [4]. The calculations show that the difference between the exact and the approximate solutions in this case does not exceed 1 per cent.

3. The fundamental problem of exterior ballistics of a projectile. The known analytic solutions of this problem were based on simplifications of either the character of the law of resistance, the hypothesis on the structure of the atmosphere or the character of projectile motion [5]. The solution of the problem under general conditions in the paper by Popov [6] is found in the form of series in powers of an artificially introduced parameter. It was assumed thereby that the density of the atmosphere is approximated by an exponential or a rational function of the elevation. V.S. Pugachev suggested a solution, obtained by the method of Poincare, in the form of a series of powers of the ballistic coefficient.

We shall show how the analytic solution of the problem regarding the motion of the mass center of the projectile may be found in a general formulation*, using the procedure expounded above.

If the vector equation of the motion of the mass center of the projectile (with zero angle of attack) is projected on the horizontal direction and on the direction normal to the trajectory, then we obtain

$$\ddot{x} = -q\cos\theta, \qquad v\dot{\theta} = -g\cos\theta \tag{3.1}$$

where $q = q(y, v)$ is the acceleration due to the force of air resistance.

* As is usual we shall assume that the earth is plane and at rest together with the atmosphere, and that the gravitational field is homogeneous.

Using two kinematic relations

$$\dot{x} = v\cos\theta, \qquad \dot{y} = v\sin\theta \tag{3.2}$$

we replace the variables x and θ by new variables w and ϕ by means of the relations

$$w = g\ln\dot{x}, \qquad \varphi = \ln\operatorname{tg}\left(\frac{\pi}{4}+\frac{\theta}{2}\right) \tag{3.3}$$

and go over in equations (3.1) and (3.2) to the argument ϕ. Then we obtain the equations of motion in the form

$$\frac{dw}{d\varphi} = q, \qquad v = \exp\left(\frac{w}{g}\right)\operatorname{ch}\varphi, \qquad \frac{dt}{d\varphi} = -\frac{v}{g}$$

$$\frac{dx}{d\varphi} = -\frac{v^2}{g\operatorname{ch}\varphi}, \qquad \frac{dy}{d\varphi} = -\frac{v^2}{g}\operatorname{th}\varphi \tag{3.4}$$

For $q = kv^2$ the first two relations in (3.4) give the equation

$$\frac{dw}{d\varphi} = k\exp\left(\frac{2w}{g}\right)\operatorname{ch}^2\varphi \tag{3.5}$$

with separable variables. The integration [7] of this equation leads to the formula which determined the velocity of the particle:

$$v = \operatorname{ch}\varphi\left\{\frac{\operatorname{ch}^2\varphi_0}{v_0^2} + \frac{k}{g}\left(\frac{1}{2}\operatorname{sh}2\varphi_0 + \varphi_0 - \frac{1}{2}\operatorname{sh}2\varphi - \varphi\right)\right\}^{-\frac{1}{2}} \tag{3.6}$$

The duration of motion and the coordinates of the particle x and y are determined from the last three equations of (3.4): (3.7

$$t = t_0 - \frac{1}{g}\int_{\varphi_0}^{\varphi} v\,d\varphi \qquad x = x_0 - \frac{1}{g}\int_{\varphi_0}^{\varphi}\frac{v^2}{\operatorname{ch}\varphi}d\varphi, \qquad y = y_0 - \frac{1}{g}\int_{\varphi_0}^{\varphi} v^2\operatorname{th}\varphi\,d\varphi$$

The solution (3.6) and (3.7), with the aid of a suitable coefficient k, for example:

$$k = cH(y_{\text{cp}})\cdot 4.74\cdot 10^{-4}c_{x_0\,\text{cp}} \tag{3.8}$$

where

$$y_{\text{cp}} = \frac{2}{3}\frac{v_0^2\operatorname{th}^2\varphi_0}{2g}, \qquad c_{x_0\,\text{cp}} = \frac{1}{2}(\min c_{x_0} + \max c_{x_0}) \tag{3.9}$$

yields a supporting function for the solution of the problem concerning projectile motion in non-homogeneous atmosphere.

In the general case the acceleration of the forces of the air-drag is expressed as:

$$q(y, v) = cH_\tau(y)\,vG(\lambda v) \tag{3.10}$$

Then from the last equation (3.7) and the first two of (3.4), follows the functional equation of the type (1.3):

$$\frac{dw}{d\varphi} - F[w] = 0$$

where (3.11)

$$F[w] = cH_\tau[v]\,vG[v], \quad v = \exp\left(\frac{w}{g}\right)\operatorname{ch}\varphi, \quad H_\tau[v] = H_\tau(y), \quad G[v] = G(\lambda v)$$

Let us find the analytical solution of the fundamental problem of exterior ballistics in the form of approximation (1.18), selecting as the supporting function

$$w^* = g \ln \frac{v^*}{\operatorname{ch} \varphi}$$

where v^* is determined by formulas (3.6) and (3.8).

Then the formulas (1.17) to (1.19) are of the form:

$$v^* = \operatorname{ch} \varphi \left\{ \frac{\operatorname{ch}^2 \varphi_0}{v_0^2} + \frac{k}{g} \left(\frac{1}{2} \operatorname{sh} 2\varphi_0 + \varphi_0 - \frac{1}{2} \operatorname{sh} 2\varphi - \varphi \right) \right\}^{-\frac{1}{2}}$$

$$y^* = y_0 - \frac{1}{g} \int_{\varphi_0}^{\varphi} v^{*2} \operatorname{th} \varphi \, d\varphi \tag{3.12}$$

$$q^* = cH_\tau(y^*) v^* G(\lambda(y^*) v^*), \qquad p = \frac{q^*}{g} n(\lambda(y^*) v^*)$$

$$v = v^* \exp \left\{ \frac{1}{g} \exp \left(\int_{\varphi_0}^{\varphi} p \, d\varphi \right) \int_{\varphi_0}^{\varphi} (q^* - kv^{*2}) \exp \left(- \int_{\varphi_0}^{\varphi} p \, d\varphi \right) d\varphi \right\}$$

The value of p is determined as follows. Since

$$\frac{F[w^*] - F[w_*]}{w^* - w_*} = \frac{c\{H(y^*) F(v^*, a^*) - H(y_*) F(v_*, a_*)\}}{g(\ln v^* - \ln v_*)}$$

then

$$p = \lim_{v_* \to v^*} \frac{c}{g} H(y^*) \frac{F(v^*, a^*) - F(v_*, a_*)}{\ln v^* - \ln v_*} = \frac{c}{g} H(y^*) \frac{dF(v^*, a^*)}{d \ln v^*} =$$

$$= \frac{n(v^*, a^*)}{g} cH(y^*) F(v^*, a^*) = \frac{n(\lambda(y^*) v^*)}{g} cH_\tau(y^*) v^* G(\lambda(y^*) v^*)$$

where $n(v, a)$ is the characteristic of air resistance:

$$n(v, a) = \frac{d \ln F(v, a)}{d \ln v}$$

and represents a well-known ballistic function.

Let us consider a concrete example. Let

$$v_0 = 562 \text{ m sec}^{-1}, \quad \theta_0 = 40^\circ, \quad c = 0.366, \quad t_0 = x_0 = y_0 = 0$$

The trajectory of the projectile calculated by the suggested method (3.12) and (3.7) is shown in Fig. 2 (Table 2), where for purposes of comparison circles indicate the results obtained by the method of numerical integration.

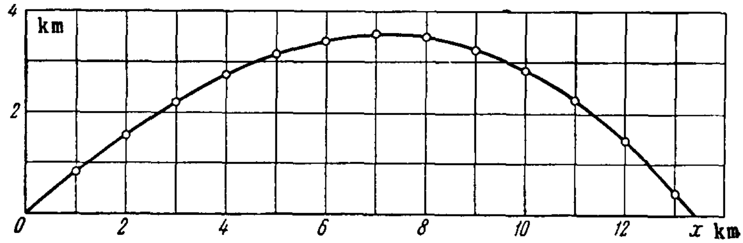

Fig. 2.

TABLE 2

Analytical Solution

φ m	x m	y m	φ m	x m	y m	φ m	x m	y m
0.763	0	0	0.463	3970	2744	−0.187	8320	3425
0,713	1050	856	0.413	4380	2920	−0.387	9480	3110
0.663	1850	1456	0.313	5130	3190	−0.587	10600	2520
0.613	2500	1890	0.263	5490	3310	−0.787	11790	1655
0 563	3050	2240	0.213	5830	3400	−0.987	12950	468
0.513	3530	2515	0.013	7130	3550	−1.187	14080	−1010

Numerical Solution

x m	y m	x m	y m	x m	y m	x m	y m
0	0	3500	2479	7000	3525	10500	2585
500	412	4000	2733	7500	3523	11000	2261
1000	809	4500	2955	8000	3479	11500	1883
1500	1189	5000	3143	8500	3392	12000	1450
2000	1548	5500	3295	9000	3260	12500	959
2500	1884	6000	3410	9500	3083	13000	405
3000	2195	6500	3487	10000	2859		

The calculations indicate that the difference in the determination of the elements of the trajectory is less than 1%. Thus, the analytical solution of the problem gives a result which practically coincides with the one obtained by the method of numerical integration of the equations of exterior ballistics of a projectile.

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